SATURATED SUBFIELDS AND INVARIANTS OF FINITE GROUPS

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Abstract. Every subfield $\mathbb{k}(\phi)$ of the field of rational functions $\mathbb{k}(x_1,\ldots,x_n)$ is contained in a unique maximal subfield of the form $\mathbb{k}(\psi)$. The element ψ is called generative for the element ϕ. A subfield of $\mathbb{k}(x_1,\ldots,x_n)$ is called saturated if it contains a generative element of each its element. We study the saturation property for subfields of invariants $\mathbb{k}(x_1,\ldots,x_n)^G$, where G is a finite group of automorphisms of the field $\mathbb{k}(x_1,\ldots,x_n)$.

1. Introduction

Consider the polynomial ring $\mathbb{k}[x_1,\ldots,x_n]$ over a field $\mathbb{k}$. Recall that a polynomial $h\in\mathbb{k}[x_1,\ldots,x_n]\setminus\mathbb{k}$ is called *closed* if the subalgebra $\mathbb{k}[h]$ is integrally closed in $\mathbb{k}[x_1,\ldots,x_n]$. For any polynomial $f\in\mathbb{k}[x_1,\ldots,x_n]\setminus\mathbb{k}$ there exist a closed polynomial h and a polynomial $F(t)\in\mathbb{k}[t]$ such that $f=F(h)$. The polynomial h is determined by the polynomial f up to affine transformations $h\to\alpha h+\beta,\ \alpha\in\mathbb{k}^\times,\beta\in\mathbb{k}$, and is called *generative* for the polynomial f.

A subalgebra $A\subseteq\mathbb{k}[x_1,\ldots,x_n]$ is said to be *saturated* if it contains generative polynomials of every its nonconstant element. Clearly, any subalgebra A which is integrally closed in $\mathbb{k}[x_1,\ldots,x_n]$ is also saturated, but the converse is not true. In the paper [1], we studied, among others, the saturation property for subalgebras of invariants $\mathbb{k}[x_1,\ldots,x_n]^G$ of finite subgroups $G\subseteq GL_n(\mathbb{k})$.

Theorem 1. [1, Thm. 2] *Let $\mathbb{k}$ be a field and $G\subseteq GL_n(\mathbb{k})$ be a finite subgroup. Then the subalgebra of invariants $\mathbb{k}[x_1,\ldots,x_n]^G$ is saturated in $\mathbb{k}[x_1,\ldots,x_n]$ if and only if the subgroup G admits no non-trivial homomorphisms $G\to\mathbb{k}^\times$.*

In particular, the saturation property for the subalgebra $\mathbb{k}[x_1,\ldots,x_n]^G$ depends on the field $\mathbb{k}$ and on the (abstract) group G, but does not depend on the matrix realization of G.

The aim of this note is to investigate the saturation property for subfields of rational invariants. We start with some definitions. Let $\mathbb{k}(x_1,\ldots,x_n)$ be the field of rational functions and $\psi\in\mathbb{k}(x_1,\ldots,x_n)\setminus\mathbb{k}$. Recall that a subfield $L\subseteq\mathbb{k}(x_1,\ldots,x_n)$ is said to be *algebraically closed* in $\mathbb{k}(x_1,\ldots,x_n)$ if every element of $\mathbb{k}(x_1,\ldots,x_n)$ which is algebraic over L belongs to L. A rational function ψ is called *closed* if the subfield $\mathbb{k}(\psi)$ is algebraically closed in $\mathbb{k}(x_1,\ldots,x_n)$. For any $\phi\in\mathbb{k}(x_1,\ldots,x_n)\setminus\mathbb{k}$ there exist a closed function ψ and an element $H(t)\in\mathbb{k}(t)$ such that $H(\psi)=\phi$. The element ψ is determined up to transformations

$$\psi\to\frac{\alpha\psi+\beta}{\gamma\psi+\delta},\quad \alpha,\beta,\gamma,\delta\in\mathbb{k},\quad \alpha\delta-\beta\gamma\neq0, \tag{1}$$

see [5, Thm. 6] or [6, Lemma 2], and is called *generative* for the element ϕ.

Generative elements appear naturally in different problems. Let $\phi=\frac{P}{Q}$ be an uncancelled form of a rational function ϕ. Consider 1-form $\omega=QdP-PdQ$.

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Its *field of constants* is a field of rational functions $\xi \in \mathbb{k}(x_1,\dots,x_n)$ satisfying $\omega \wedge d\xi = 0$. It is known that this field is generated by the generative element ψ: for two variables it goes back to classical works of A. Poincaré on algebraic integration of differential equations, and the general case is discussed in [5]. Further, if the kernel of a $\mathbb{k}$-derivation of the field $\mathbb{k}(x_1,\dots,x_n)$ is one-dimensional, then it has a form $\mathbb{k}(\psi)$ for some closed rational function ψ. Conversely, in [3] for any closed rational function ψ a $\mathbb{k}$-derivation of the field $\mathbb{k}(x_1,\dots,x_n)$ with the kernel $\mathbb{k}(\psi)$ is constructed. Some characterizations of closed rational functions are obtained in [2] and [6]. In [2], an analogue of Stein's theorem for rational functions is proved and an estimate of the number of reducible fibers is obtained. Finally, in [5, Sec. 7] an algorithm that finds a generative element ψ for a given rational function ϕ is given.

A subfield $L \subseteq \mathbb{k}(x_1,\dots,x_n)$ is said to be *saturated* if for any $\phi \in L \setminus \mathbb{k}$ the generative function ψ of ϕ is contained in L. Clearly, every algebraically closed subfield of $\mathbb{k}(x_1,\dots,x_n)$ is saturated. The results below imply that the converse is not true.

Let G be a finite group of automorphisms of the field $\mathbb{k}(x_1,\dots,x_n)$. We study the saturation property for the subfield of invariants $\mathbb{k}(x_1,\dots,x_n)^G$. It is interesting to remark that, as follows from Lüroth's Theorem, the subfield $\mathbb{k}(x_1,\dots,x_n)^G$ is saturated in $\mathbb{k}(x_1,\dots,x_n)$ if and only if every one-dimensional G-invariant subfield of $\mathbb{k}(x_1,\dots,x_n)$ lies in $\mathbb{k}(x_1,\dots,x_n)^G$. Geometrically this means that every G-equivariant rational morphism from $\mathbb{k}^n$ to a curve is G-invariant.

2. Main results

The next result yields a sufficient condition for the subfield of invariants to be saturated.

Theorem 2. *Let $\mathbb{k}$ be a field of characteristic zero and G be a finite group of automorphisms of the field $\mathbb{k}(x_1,\dots,x_n)$. Suppose that G has neither nontrivial abelian factor groups nor factor groups which are isomorphic to the alternating group A_5. Then the subfield $\mathbb{k}(x_1,\dots,x_n)^G$ is saturated in $\mathbb{k}(x_1,\dots,x_n)$.*

Proof. Suppose that the generative function ψ of an element $\phi \in \mathbb{k}(x_1,\dots,x_n)^G \setminus \mathbb{k}$ is not invariant. It follows from (1) that the element ψ is determined by ϕ up to the action of the group $\mathrm{PGL}_2(\mathbb{k})$. Therefore the group G admits a non-trivial homomorphism to $\mathrm{PGL}_2(\mathbb{k})$. This homomorphism is defined over a finitely generated extension of the field of rational numbers. Any such extension is isomorphic to a subfield of the field of complex numbers $\mathbb{C}$. On the other hand, from the classification of finite subgroups in $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ (see, for example, [7, 4.4]) it follows that every such subgroup is either solvable or isomorphic to the group A_5. This contradiction completes the proof of the theorem. □

Example 1. The automorphism group of the field $\mathbb{C}(x)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. Let $G \subset \mathrm{PSL}_2(\mathbb{C})$ be the group of rotations of icosahedron, which is isomorphic to A_5. Then the subfield $\mathbb{C}(x)^G$ is not saturated in $\mathbb{C}(x)$.

Now we consider subfields of invariants which correspond to (regular) representations of a finite group G. Note that the group $\mathrm{PSL}_2(\mathbb{C})$ contains a unique (up to conjugation) subgroup isomorphic to A_5. Denote its preimage in $\mathrm{SL}_2(\mathbb{C})$ by I_{120}. This is a group of order 120. Since A_5 admits no non-trivial two-dimensional representations, the subgroup I_{120} has no subgroups isomorphic to A_5. The center Z of I_{120} consists of two elements, and $I_{120}/Z \simeq A_5$. Therefore the group I_{120} coincides with its commutator subgroup.

Theorem 3. *Let $\mathbb{k}$ be an algebraically closed field of characteristic zero and $G \subset \mathrm{GL}_n(\mathbb{k})$ be a finite subgroup. The following conditions are equivalent:*

(i) the group G has neither nontrivial abelian factor groups nor factor groups isomorphic to I_{120};

(ii) the subfield $\mathbb{k}(x_1,\dots,x_n)^G$ is saturated in $\mathbb{k}(x_1,\dots,x_n)$.

Proof. (i)⇒(ii) Suppose that there exists an element $\phi \in \mathbb{k}(x_1,\dots,x_n)^G \setminus \mathbb{k}$ whose generative element $\psi = \frac{p}{q},\ p,q \in \mathbb{k}[x_1,\dots,x_n]$ is not invariant. Since the subfield $\mathbb{k}(\psi)$ is G-invariant, so is the subspace generated by p and q. Thus the group G admits a nontrivial homomorphism to the group $\mathrm{GL}_2(\mathbb{k})$. By assumptions, the image of G is contained in the subgroup $\mathrm{SL}_2(\mathbb{k})$. Further, the image of G in $\mathrm{Aut}(\mathbb{k}(\psi))$ should be a subgroup isomorphic to A_5. Therefore the image of G in $\mathrm{SL}_2(\mathbb{k})$ coincides with I_{120}, which yields a contradiction.

(ii)⇒(i) If the group G has a nontrivial abelian factor group, then the subalgebra $\mathbb{k}[x_1,\dots,x_n]^G$ is not saturated in $\mathbb{k}[x_1,\dots,x_n]$ (Theorem 1). Hence the subfield $\mathbb{k}(x_1,\dots,x_n)^G$ is also not saturated in $\mathbb{k}(x_1,\dots,x_n)$. In the case when G admits a surjective homomorphism onto the group I_{120} we need an auxiliary lemma. It is a well-known statement from representation theory, but having no reference we give a short proof.

Lemma 1. *Let $\mathbb{k}$ be a field of characteristic zero and $G \subset \mathrm{GL_n}(\mathbb{k})$ be a finite subgroup. Then every irreducible representation of the group G can be realized as a subrepresentation in the G-module $\mathbb{k}[x_1,\dots,x_n]$.*

Proof. Since the subspace of g-fixed vectors in $\mathbb{k}^n$ is proper for every $g \in G,\ g \neq 1$, there exists a vector $v \in \mathbb{k}^n$ for which the mapping $\gamma_v : G \to \mathbb{k}^n,\ \gamma_v(g) = gv$ is injective. Let $F(G)$ be the space of $\mathbb{k}$-valued functions on the group G with the canonical structure of G-module. The natural homomorphism

$$\gamma_v^\star : \mathbb{k}[x_1,\dots,x_n] \to F(G), \quad (\gamma_v^\star)(g) = f(\gamma_v(g))$$

is G-equivariant and surjective. On the other hand, the G-module $F(G)$ is dual to the group algebra of the group G and therefore every simple G-module is contained in $F(G)$ with nonzero multiplicity. □

Consider a G-submodule $U = \langle p,q\rangle$ in $\mathbb{k}[x_1,\dots,x_n]$ which is isomorphic to the simple two-dimensional I_{120}-module.

Lemma 2. *Let $\mathbb{k}$ be a field and $H \subseteq \mathrm{GL}_2(\mathbb{k})$ be a finite subgroup containing a non-scalar matrix. Then the subfield $\mathbb{k}(y_1,y_2)^H$ is not saturated in $\mathbb{k}(y_1,y_2)$.*

Proof. Consider the finite set of elements $\{\frac{g\cdot y_1}{g\cdot y_2} : g \in G\}$. At least one of elementary symmetric polynomials σ_i of these elements is non-constant, and a generative element for $\sigma_i \in \mathbb{k}(y_1,y_2)^H$ is $\frac{y_1}{y_2} \notin \mathbb{k}(y_1,y_2)^H$.

□

Applying Lemma 2 to the image H of G in GL(U), we get the statement. □

Corollary 1. *Let $G \subset \mathrm{GL}_n(\mathbb{k})$ be a subgroup isomorphic to A_5. Then the subfield $\mathbb{k}(x_1,\dots,x_n)^G$ is saturated in $\mathbb{k}(x_1,\dots,x_n)$.*

It follows from Corollary 1 and Example 1 that in the case of subfields of invariants the saturation property depends not only on the field $\mathbb{k}$ and the group G, but also on the representation of G by means of automorphisms of the field $\mathbb{k}(x_1,\dots,x_n)$.

Since the group A_5 admits a faithful three-dimensional representation, Corollary 1 allows for $n \geq 3$ to construct a saturated subfield in $\mathbb{k}(x_1,\dots,x_n)$ with $\mathbb{k}(x_1,\dots,x_n)$ being algebraic over this subfield. Clearly, a subfield $\mathbb{k} \subset L \subseteq \mathbb{k}(x_1)$ is saturated if and only if $L = \mathbb{k}(x_1)$. It remains to consider the case $n = 2$. Lemma 2 and Theorem 1 imply that for a proper finite subgroup $G \subset \mathrm{GL}_2(\mathbb{k})$ the

subfield of invariants $\Bbbk(x_1, x_2)^G$ in not saturated. Nevertheless, there is a saturated subfield of invariants in $\Bbbk(x_1, x_2)$. As shown in [4], see also [8, p. 78], there is a subgroup F of order 1080 in the group $\mathrm{SL}_3(\Bbbk)$ such that the factor group F/Z, where Z is the center of $\mathrm{SL}_3(\Bbbk)$, is isomorphic to the alternating group A_6. This defines an effective action of the group A_6 on the projective plane $\mathbb{P}^2$ and on the field $\Bbbk(x_1, x_2)$. By Theorem 2, the subfield $\Bbbk(x_1, x_2)^{A_6}$ is saturated in $\Bbbk(x_1, x_2)$.

Finally, we give three examples which show that saturation of the subalgebra of invariants does not imply saturation of the field of invariants.

Example 2. Consider the following subgroups:

- $G = I_{120} \subset \mathrm{GL}_2(\mathbb{C})$;
- $G = \langle \theta \rangle \subset \mathrm{GL}_2(\mathbb{R})$, where θ is the 120°-rotation of $\mathbb{R}^2$;
- $G = \langle \tau \rangle \subset \mathrm{GL}_2(\Bbbk)$, where $\tau(x_1) = x_2$, $\tau(x_2) = x_1$, and $\Bbbk$ is a field of characteristic two.

In each of these cases the subalgebra $\Bbbk[x_1, x_2]^G$ is saturated in $\Bbbk[x_1, x_2]$ (Theorem 1), but the subfield $\Bbbk(x_1, x_2)^G$ is not saturated in $\Bbbk(x_1, x_2)$ (Lemma 2).

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